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Surface-pressure fluctuations induced by boundary-layer *flow at finite Mach number*

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A theory describing boundary-layer surface-pressure fluctuations on a rigid surface is presented in a form that illustrates the main effect of compressibility. The most significant effect is that the correlation area is proportional to the square of mean-flow Mach number so it does not vanish in flow of finite compressibility. Modifications of the wave-number and frequency spectra by this effect are described, and the results applied to the computation of large plate response. That computation incorporates the effect of fluid loading, which enters the response equations as a dissipative term for components at supersonic phase velocity but merely as an added loading for subsonic components.

1. Introduction

This paper is concerned with a study of two effects which are frequently regarded as negligible in discussions of boundary-layer noise. The first is the manner in which the spectral-energy distribution is subject to modification by finite-Mach-number effects. The second is concerned with details of structural response to boundary-layer turbulence, and its susceptibility to changes in mean-flow Mach number. It is known that the intensity of the turbulence-induced surface pressure is a slowly varying function of Mach number. This point, evident from experimental studies (Willmarth & Wooldridge 1962, Hodgson 1962, Kistler & Chen 1962), is explained by Lilley (1963) to be due, very largely, to the fact that, even in high Mach-number flows, the region of turbulence giving rise to the dominant part of the surface pressure, being relatively close to the surface, is at a considerably lower velocity. He showed how arguments based on incompressible flow equations need be modified only slightly at higher Mach numbers. In general, that conclusion is undoubtedly true, but it would be quite misleading to regard it as the case for all spectral components of the flow.

Even in boundary layers formed at very low speeds some spectral components are associated with supersonic phase velocities and must be subject to modification by compressibility effects. The correlation area is a parameter critically dependent on these effects. Incompressible flow arguments (Kraichnan 1956, Phillips 1956, Hodgson 1962, Ffowcs Williams & Lyon 1963) have shown the area to be zero, with the instantaneous net force applied by pressure fluctuations on the boundary also being zero. This is true irrespective of the temporal variation in surface pressure. Obviously this result is irrelevant whenever incompressible flow equations prove inadequate. That is generally the case, for the result may be re-expressed as a statement that the power spectral density at zero wave-number is zero irrespective of the frequency. Any finite frequency associated with zero wave-number implies, not only a supersonic phase velocity, but an infinite one, so that the incompressible solution should be strongly qualified. In fact there is a net force and consequently a finite correlation area, but proportional to the square of mean-flow Mach number. It will be argued that, even in low Mach number flows, this compressible-flow effect will dominate the low wave-number region of the wall-pressure spectrum, and will modify previous conclusions that the spectrum will increase from zero quadratically with increasing wave-number. A wave-number limit, below which proper deductions must be based on compressible-flow equations will appear, and that limit is suggested to be of the order M/δ^*, M being the mean flow Mach number and δ^* the displacement thickness.

This modification of the wall-pressure spectrum will have a direct bearing on surface response in particular spectral ranges. Those ranges are the ones directly responsible for the transmission of sound by structures, especially when the panel size is large. Response calculations will be described to show how surface vibration is also modified by compressibility effects, in precisely the same ranges as is the forcing spectrum. At supersonic phase velocities, compressibility effects are important and sound waves become possible. The sound field can then act as an energy drain to the structure and appears in the response equations as an added damping term. However, at subsonic phase speeds, compressibility effects are negligible and the flow may be regarded as incompressible. An added loading is still evident, but represents no energy drain and affects the response level only inasmuch as it changes the resonance frequency. Mechanical damping thus provides the only control of the resonant response.

2. The surface-pressure spectrum at finite Mach number

The object of this study is to bring out the influence of fluid compressibility on the spectrum of surface-pressure fluctuations induced by turbulent flow. Viscous effects are of no particular concern, so we neglect them, but assume, of course, that they play their full role in bringing about and maintaining a turbulent state. Then, the exact equations of motion can be written in a form where the surface pressure on a rigid boundary is given by a volume integral of a turbulent-source term,

$$\frac{\partial^2 T_{ij}}{\partial z_i \partial z_j}(\mathbf{z}, t).$$

T_{ij} is Lighthill's (1952) turbulence stress tensor, $\mathbf{z}$ is a position vector and t is time. z_i is a tensor where repeated suffices are to be summed over 1, 2 and 3. The direction parallel with the 2 axis will be that normal to the indefinitely large flat surface lying in the 1, 3 plane. The surface pressure is designated $p(\mathbf{y}, t)$, $\mathbf{y}$ being a position vector in the surface

$$p(\mathbf{y}, t) = \frac{1}{2\pi}\int_{\Gamma} \frac{\partial^2 T_{ij}}{\partial z_i \partial z_j}\left(\mathbf{z}, t - \frac{r}{a_0}\right)\frac{d\mathbf{z}}{r}, \qquad (2.0)$$

where a_0 is the speed of sound in the stagnation flow and r is the distance separating a particular source point $\mathbf{z}$ from the position where the pressure is being measured $\mathbf{y}, r = |\mathbf{z}-\mathbf{y}|$.

The three-dimensional power spectrum of wall pressure can be formed very easily once the generalized Fourier transform of the pressure is known. This transform we shall denote $\phi(\mathbf{k},\omega)$, $\mathbf{k}$ being a wave-number vector in the plane associated with the surface, and ω an angular frequency, i.e.

$$\phi(\mathbf{k},\omega) = \iint p(\mathbf{y},t)\, e^{-i\mathbf{k}.\mathbf{y}}\, e^{-i\omega t}\, d\mathbf{y}\, dt. \tag{2.1}$$

The wall pressure is given by the inverse of this equation

$$p(\mathbf{y},t) = \frac{1}{(2\pi)^3}\iint \phi(\mathbf{k},\omega)\, e^{i\mathbf{k}.\mathbf{y}}\, e^{i\omega t}\, d\mathbf{k}\, d\omega. \tag{2.2}$$

In a similar way we define $W_{ij}(\mathbf{z},\omega)$ to be the generalized Fourier time transform of the turbulence stress tensor, so that

$$T_{ij}\left(\mathbf{z}, t-\frac{r}{a_0}\right) = \frac{1}{2\pi}\int W_{ij}(\mathbf{z},\omega)\exp\left\{i\omega\left(t-\frac{r}{a_0}\right)\right\} d\omega. \tag{2.3}$$

The equation for the Fourier transform of surface pressure can then be written

$$\phi(\mathbf{k},\omega) = \iint \frac{1}{2\pi}\int_V \frac{\partial^2 T_{ij}}{\partial z_i \partial z_j}\left(\mathbf{z}, t-\frac{r}{a_0}\right)\frac{d\mathbf{z}}{r}\, e^{-i\mathbf{k}.\mathbf{y}}\, e^{-i\omega t}\, d\mathbf{y}\, dt, \tag{2.4}$$

$$= \frac{1}{(2\pi)^2}\iiiint_V \frac{\partial^2 W_{ij}}{\partial z_i \partial z_j}(\mathbf{z},\omega^*)\exp\left\{i\omega^*\left(t-\frac{r}{a_0}\right)\right\} d\omega^* \frac{d\mathbf{z}}{r}\, e^{-i\mathbf{k}.\mathbf{y}}\, e^{-i\omega t}\, d\mathbf{y}\, dt. \tag{2.5}$$

The integration over time and frequency ω^* is straightforward and leads to a result equating the frequencies in the pressure and source terms

$$\phi(\mathbf{k},\omega) = \frac{1}{2\pi}\iint_V \frac{\partial^2 W_{ij}}{\partial z_i \partial z_j}(\mathbf{z},\omega)\, e^{-i\omega(r/a_0)}\frac{d\mathbf{z}}{r}\, e^{-i\mathbf{k}.\mathbf{y}}\, d\mathbf{y}. \tag{2.6}$$

The surface integral over $\mathbf{y}$ is also relatively straightforward provided the surface is sufficiently large that the result is independent of the surface shape, and this we shall assume to be the case.

$$\frac{1}{2\pi}\int e^{-i\omega r/a_0}\, e^{-i\mathbf{k}.\mathbf{y}}\frac{d\mathbf{y}}{r} = -\frac{i\exp\left[-iz_2\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}\right]}{\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}}\, e^{-i\mathbf{k}.\mathbf{z}}, \tag{2.7}$$

where k is written for $|\mathbf{k}|$. Equation (2.6) can then be rewritten as

$$\phi(\mathbf{k},\omega) = -i\int_V \frac{\partial^2 W_{ij}}{\partial z_i \partial z_j}(\mathbf{z},\omega)\,\frac{\exp\left[-iz_2\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}\right]}{\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}}\, e^{-i\mathbf{k}.\mathbf{z}}\, d\mathbf{z}. \tag{2.8}$$

The stress tensor together with its first derivative can be assumed zero on the surface bounding the volume V. They are zero on the rigid boundary, and the other boundaries can be considered sufficiently distant from the source as not to affect the result. This step is justified by the fact that the influence of localized turbulence takes a finite time to travel to remote positions. Any distant control surface could then be assumed beyond the region currently subjected to that influence. A complication enters when the limit of incompressible flow is approached by allowing the propagation speed to become infinite. However, it can easily be shown that the surface source terms are then of the order (r^{-4}) at large r, so that their integral remains zero. Then

$$\phi(\mathbf{k},\omega) = \int_V W_{ij}(\mathbf{z},\omega)\,[\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}\,\delta_{i2}+k_i]\,[\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}\,\delta_{j2}+k_j] \times \frac{i\exp[-iz_2\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}]}{\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}}\,e^{-i\mathbf{k}.\mathbf{z}}\,d\mathbf{z}. \tag{2.9}$$

This equation can be written in terms of the three-dimensional generalized Fourier transform of a source function $\theta_{ij}(z_2,\mathbf{k},\omega)$, say, by carrying out the integration over a surface parallel to the plane boundary

$$\theta_{ij}(z_2,\mathbf{k},\omega) = \int_{s(z_2=\text{const.})} W_{ij}(\mathbf{z},\omega)\,e^{-i\mathbf{k}.\mathbf{z}}\,ds,$$

$$\phi(\mathbf{k},\omega) = \int_0^\infty \theta_{ij}(z_2,\mathbf{k},\omega)\,[\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}\,\delta_{i2}+k_i]\,[\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}\,\delta_{j2}+k_j] \times \frac{i\exp[-iz_2\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}]}{\{(\omega/a_0)^2-k^2\}^{\frac{1}{2}}}\,dz_2. \tag{2.10}$$

The power spectrum of the wall pressure is formed by multiplying this equation by its complex conjugate, but we shall not do that, since the expression becomes somewhat more bulky without making its form any simpler. We can examine the properties of the spectrum at low frequencies and low wave-numbers, which is our particular concern here, on the basis of the above expression, remembering of course that the spectrum depends on $\phi(\mathbf{k},\omega)$ quadratically.

It is clear from equation (2.10) that, for any wave-number k, there exists a critical frequency above which the wall pressure is part of an oscillatory field, or sound wave. Below that frequency, the pressure decays exponentially from source and the compressibility of the fluid is then of no great consequence in setting its strength. Conditions at infinity must be used to ensure outgoing waves in the high-frequency solution and to ensure a finite contribution in the second solution. We write the limiting form of these two cases separately. First,

$$\phi(\mathbf{k},\omega)_{|\omega|\gg a_0|\mathbf{k}|} = \int_0^\infty \theta_{ij}(z_2,\mathbf{k},\omega)\,\{(\omega/a_0)\,\delta_{i2}+k_i\}\,\{(\omega/a_0)\,\delta_{j2}+k_j\} \times \frac{i\exp\{-iz_2(\omega/a_0)\}}{\omega/a_0}\,dz_2. \tag{2.11}$$

This solution is entirely relevant at $k = 0$, having the form

$$\phi(0,\omega) = \int_0^\infty \theta_{22}(z_2,0,\omega)\,i\frac{\omega}{a_0}\exp\left\{-iz_2\frac{\omega}{a_0}\right\}dz_2, \tag{2.12}$$

an expression that shows how the surface integral of wall pressure is entirely determined by the sound radiated by the turbulent flow in a direction perpendicular to the surface; an exact restatement of the Fourier transform of equation (2.12) being

$$\frac{1}{2\pi}\int \phi(0,\omega)\, e^{i\omega t}\, d\omega = \int_s p(\mathbf{y},t)\, d\mathbf{y},$$

$$= \frac{1}{2\pi}\iint_0^\infty \theta_{22}(z_2, 0, \omega)\frac{i\omega}{a_0}\exp\left\{i\omega\left(t-\frac{z_2}{a_0}\right)\right\} d\omega\, dz_2,$$

$$\int_s p(\mathbf{y},t)\, d\mathbf{y} = \frac{1}{a_0}\int_V \frac{\partial T_{22}}{\partial t}(\mathbf{z}, t - z_2/a_0)\, d\mathbf{z},$$

$$\int_s \frac{\partial p}{\partial t}(\mathbf{y},t)\, d\mathbf{y} = \frac{1}{a_0}\int_V \frac{\partial^2 T_{22}}{\partial t^2}\{z, t-(z_2/a_0)\}\, dz. \tag{2.13}$$

The first major influence of compressibility thus emerges to qualify the frequently quoted result that the wave-number spectral density approaches zero at low wave-numbers, or that the instantaneous surface force vanishes. It does in incompressible flow, but finite compressibility ensures a finite value.

This finding has an interesting corollary in two related studies. The first is that the final expression in equation (2.13) relates two terms that appear in a particular application of Curle's (1955) extension to Lighthill's theory of aerodynamic sound. These terms have in the past been regarded to be of differing orders of magnitude, an impression that is corrected only when the small, but finite, fluid compressibility is taken into account. However, this aspect has already been clarified by Powell (1960). The second relates to experimental studies of surface-pressure-correlation areas in boundary-layer flows. The theory shows that, in this instance, the correlation area is determined by what is effectively the turbulence sound field. It must, therefore, be subjected to change by any alteration in the *acoustic* environment. Consequently, that environment should be subject to the same careful control as in the aerodynamic flow, in experimental studies of correlation areas.

From equation (2.11) it is clear that the power spectral density in the régime, $|\omega| \gg a_0|\mathbf{k}|$, approaches zero with frequency squared at low frequencies. $\theta_{ij}(z_2, \mathbf{k}, \omega)$, being the transform of a source term depending quadratically on velocity fluctuations, will incorporate 'difference tones' which are bound to generate a flat spectrum at low frequencies. The form of the pressure spectrum then follows immediately from the wave number, or frequency, weighting functions, ω/a_0 in this case leading to ω^2/a_0^2 in the quadratic spectrum function.

In the low-frequency régime, where $|\omega| \ll a_0|\mathbf{k}|$, the usual incompressible-flow arguments apply and equation (2.10) reduces to the equation familiar to that problem:

$$\phi(\mathbf{k},\omega)_{|\omega| \ll a_0|\mathbf{k}|} = \int_0^\infty \theta_{ij}(z_2, \mathbf{k}, \omega)\{k\delta_{i2}+k_i\}\{k\delta_{j2}+k_j\}\frac{e^{-kz_2}}{k}\, dz_2. \tag{2.14}$$

At zero frequency this equation is invariably valid so that it defines precisely the power spectrum along the wave-number axis:

$$\phi(\mathbf{k},0) = \int_0^\infty \theta_{ij}(z_2, \mathbf{k}, 0)\{k\delta_{i2}+k_i\}\{k\delta_{j2}+k_j\}\frac{e^{-kz_2}}{k}\, dz_2. \tag{2.15}$$

There are two points to notice about this result. First, contributions to the surface pressure come from all regions of the boundary layer, the bigger the scale of pressure (i.e. the lower the wave-number), the smaller will be the effect of the weighting term tending to localize the source region, i.e. $\exp(-kz_2) \to 1$ as $k \to 0$. Secondly, the pressure transform $\phi(\mathbf{k}, \omega)$ is proportional to k times some integral function of the source transform $\theta_{ij}(z_2, \mathbf{k}, \omega)$. $\phi(\mathbf{k}, \omega)$ must then increase from zero in direct proportion to k at low enough wave-number, its value at $k = 0$, determining the wave-number spectral density there, being zero.

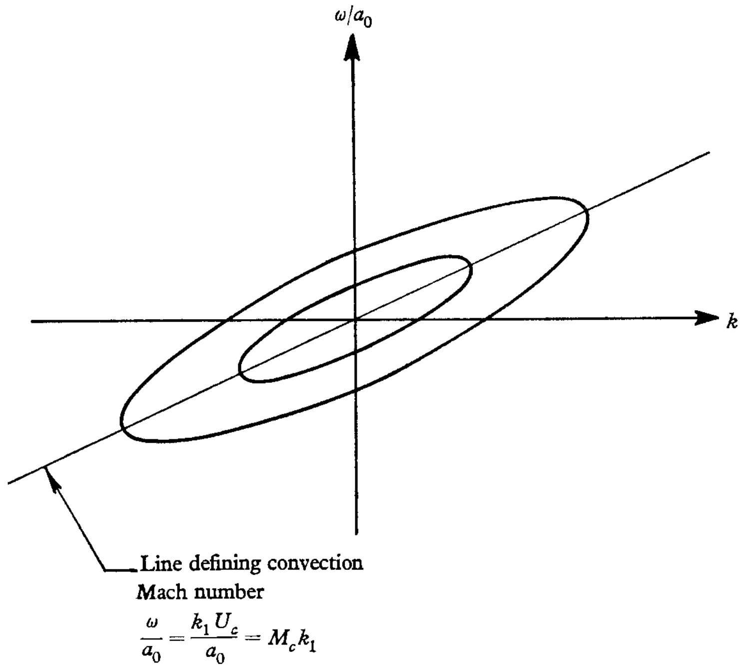

FIGURE 1. Contours of constant spectral level in a typical source spectrum of the boundary-layer flow.

Another important point to note from equation (2.14) is that the equation has very little to say about the form of the frequency spectrum of wall pressure. It says no more in fact than that the contribution from a particular source region will have precisely the same frequency spectrum as does the source in that region. However, that information is very significant because the sources in turbulent flow are convected by the mean motion so the frequency spectrum is, in the main, merely a reproduction of the downstream wave-number spectrum; i.e. $\theta_{ij}(z_2, \mathbf{k}, \omega - k_1 U_c)$ tends to vary slowly with changes in k, U_c being the local convection velocity. It follows that the wall-pressure spectrum will also display this convective property, but for a large variety of convection speeds.

We are now in a position to discuss general properties of the pressure spectrum at both low frequencies and wave-numbers, and to examine the way compressibility alters the previous incompressible-flow arguments. Our discussion must be based on typical features to be expected in the source spectrum to which $\theta_{ij}(z_2, \mathbf{k}, \omega)$ is related. We shall assume that spectrum to be typically of the form sketched out in figure 1, a form characteristic of convective fields. Our foregoing discussion shows how the wall-pressure spectrum is composed of a superposition of spectra of these types weighted according to their position. Our major interest is in the way they are weighted with respect to frequency and wave-number, and that follows from equations (2.12) and (2.15) dealing with the compressible and incompressible régimes respectively.

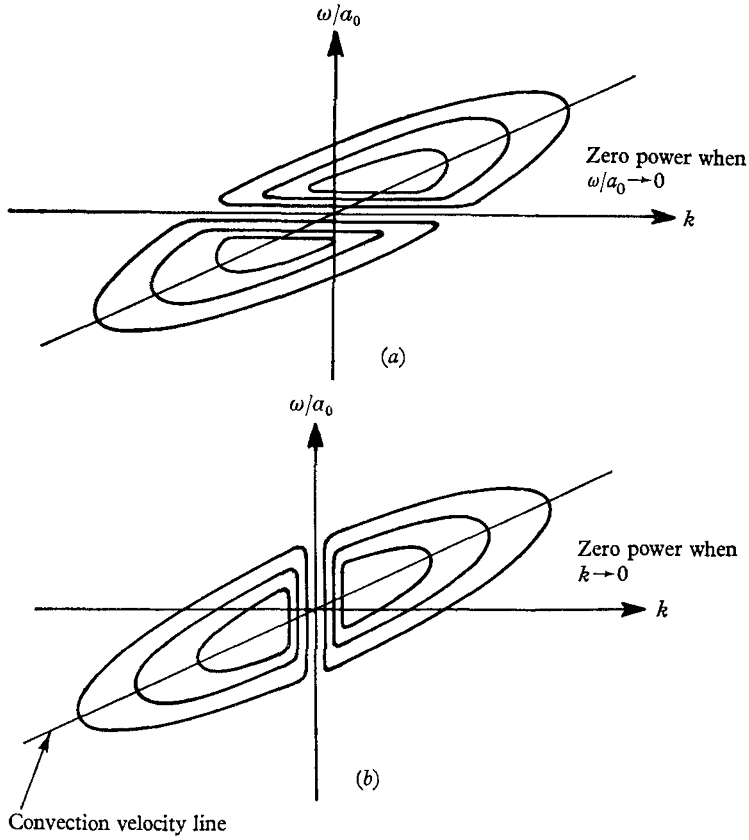

FIGURE 2. Basic form of the surface-pressure spectrum. (a) Compressible régime, ω^2/a_0^2 weighting; (b) incompressible régime, k^2 weighting.

Figure 2 illustrates the wall-pressure spectrum to be expected in the two extreme ranges. The first is that where compressibility is always important so that $|\omega| > a_0|\mathbf{k}|$. The spectrum is characterized by its approach to zero at zero frequency. The second limiting case is the more familiar one relevant to entirely incompressible flows so that $|\omega| < a_0|\mathbf{k}|$. This spectrum is characterized by its approach to zero at zero wave-number. Both spectra display the convective features characteristic of the source spectrum in figure 1.

Previous discussions of the wall-pressure spectra have been concerned with entirely incompressible flow, so that figure 2 (*b*) illustrates the basis for their conclusion that the wave-number spectrum approaches zero as k^2 at low wave-number. An application of Taylor's hypothesis of rigid convection relates this to the frequency spectrum, which would display a level approaching zero with frequency squared at low frequency. However, it is evident that both the assumption of incompressible flow and of rigid convection break down at low values of wave-number. The modification of the model is, however, relatively simple. The compressible-flow spectrum will display features of figure 2 (*a*) in the high-frequency régime where $|\omega| \gg a_0|\mathbf{k}|$, but that of figure 2 (*b*) at low frequencies. That spectrum is then generated by a suitable superposition of figure 2(*a*) and (*b*) and is sketched out in figure 3. The main point to note here is that the spectral

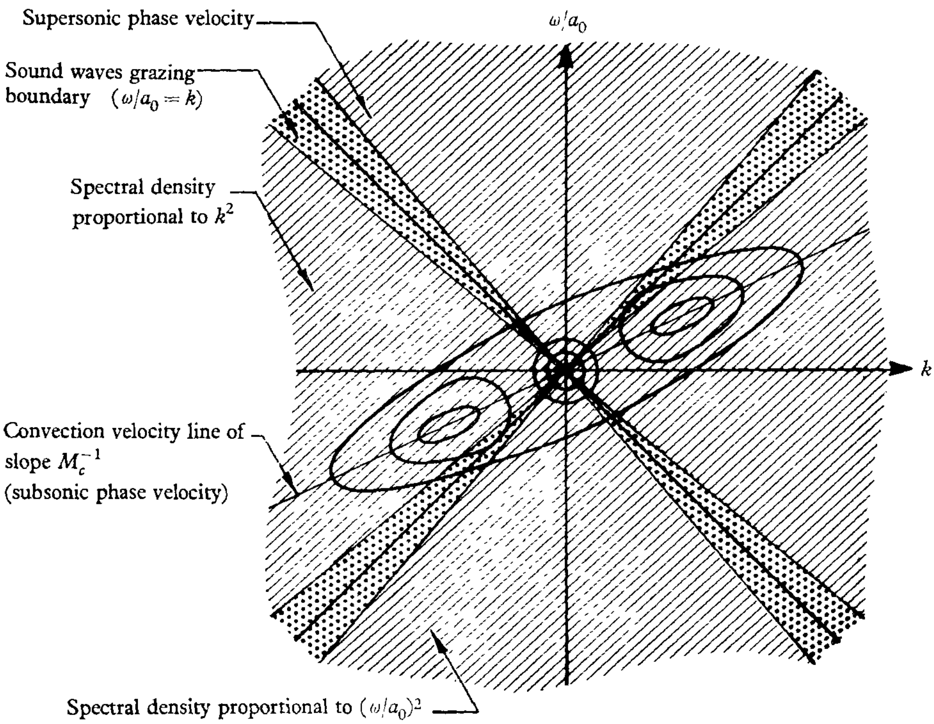

FIGURE 3. A typical surface-pressure power spectrum in flow of finite compressibility. The example chosen is for a predominantly subsonic flow where most of the energy is naturally concentrated close to the origin of the k_3 wave-number component, i.e. $k \approx k_1$.

density is now zero only when both frequency and wave-number are zero. At higher frequencies or wave-numbers, however, the dominant feature remains that of convection and the two spectra are again likely to be related through Taylor's hypothesis of rigid convection.

In practice, two-dimensional spectra are rarely measured so that the consequences of compressibility effects on the simpler spectral functions are of more direct interest. We deal first with the wave-number spectrum $P(k)$, say, which is obtained by integrating the two-dimensional spectrum over all frequencies. Its asymptotic form at low wave-number is easily deduced from an integration of equation (2.12):

$$\int \phi(0, \omega)\, d\omega = \iint_0^\infty \theta_{22}(z_2, 0, \omega) \frac{i\omega}{a_0} \exp\left(-iz_2 \frac{\omega}{a_0}\right) d\omega\, dz_2. \tag{2.16}$$

θ_{ij} is a three-dimensional Fourier transform of the source function T_{ij} which increases in direct proportion to a typical dynamic head $\bar{\rho}U^2$. Frequencies tend to have the Strouhal proportionality $\omega \sim (U/\delta^*)$, where δ^* is the boundary-layer-displacement thickness, regarded here as the typical scale of the turbulent field. These values impose a definite proportionality on the function θ_{ij}:

$$\theta_{ij} \sim \bar{\rho}U\delta^{*3}. \tag{2.17}$$

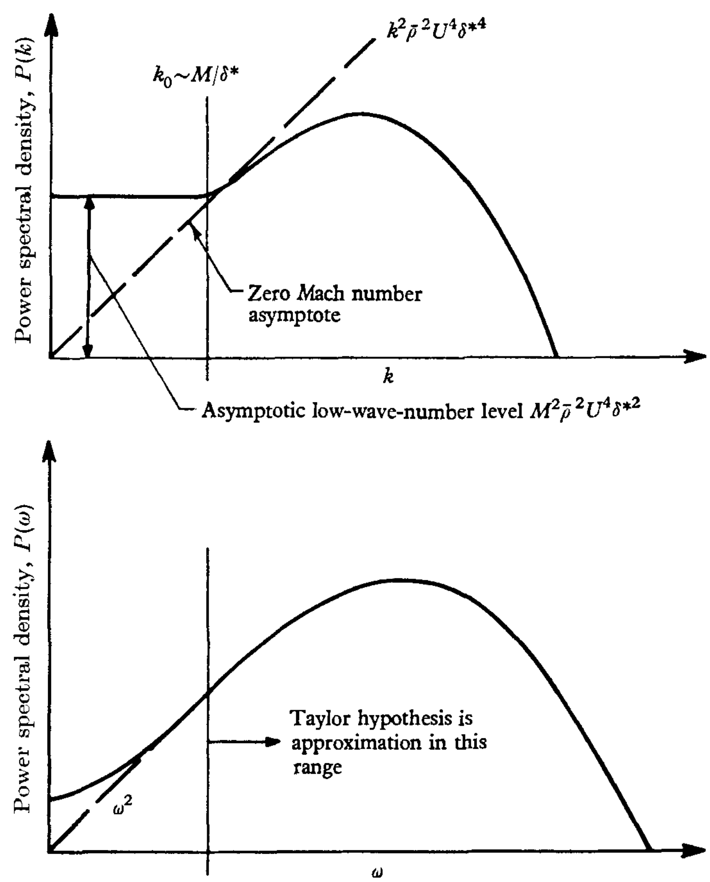

FIGURE 4. General features of wave-number and frequency spectra of the surface-pressure field induced by a turbulent boundary layer at finite Mach numbers.

The dimensional form of equation (2.16) is then $\bar{\rho}(U^3/a_0)\,\delta^{*2}$, leading to an asymptotic low-wave-number limit, $P(k \to 0)$, proportional to the square of the mean-flow Mach number $M = (U/a_0)$

$$\begin{aligned}\lim_{k\to 0} P(k) &\sim \bar{\rho}^2\{U/a_0\}^2\, U^4\delta^{*2}, \\ &\sim M^2\bar{\rho}^2U^4\delta^{*2}.\end{aligned} \tag{2.18}$$

At higher wave numbers the effects of compressibility are less important, and the spectrum will display features directly attributable to incompressible-flow phenomena. The power may rise with wave-number squared above the compressible-flow-dominated wave-number range. A typical spectrum is sketched out in figure 4(*a*) and represents the integral over all frequencies of the more general spectrum illustrated in figure 3. The example shown is for subsonic flow. At higher Mach numbers the plateau at low wave-numbers may obscure any portion increasing with increasing values of k. For a considerable part of the low-wave-number range, incompressible-flow arguments would then be inadequate. The dimensional analysis applied to the spectrum based on equation (2.14) shows how the level increases in proportion to $k^2\bar{\rho}^2U^4\delta^{*4}$, so that the incompressible-flow level equals the asymptotic level, where compressible effects are dominant, at a wave-number k_0, where

$$k_0\delta^* \sim M. \tag{2.19}$$

This limiting value is also sketched out in figure 4, but is valid only as long as the asymptotic low-wave-number level is lower than the peak level, a condition limited by the mean-stream Mach number.

In a similar way the frequency spectrum may be computed by integrating the spectrum of figure 3 over all wave-numbers. In this case there appears to be no good reason why the spectrum should be zero at a very low frequency, since a finite power seems to exist there, at least over some wave numbers. In low-Mach-number flows, the compressibility seems to play no significant role in setting the low-frequency spectral level, a point evident from figure 3, where the energy is mainly concentrated in the region described well by incompressible-flow arguments.

The suggestion put over by Hodgson (1962), that the spectrum tends to vanish at low frequency, based as it is on incompressible-flow arguments and an application of Taylor's hypothesis, can only be valid away from the region of very low frequency. However, Hodgson's measurements show that region to be extensive, and that the bulk of the spectrum demonstrates the convection of the incompressible motion where the wave-number spectrum approaches zero like k^2 at low wave-number. A sketch of such a spectrum is given in figure 4(*b*), although the current study throws no new light on its form apart from the prediction of a non-zero asymptote at low enough frequency.

Before going on to describe some of the effects of these conclusions on the response of an elastic boundary to turbulent flow, it is worth summarizing the basic results. In incompressible flow the wave-number spectrum approaches zero like k^2. The frequency spectrum being, in the main, a demonstration of convective effects, reproduces this form and tends to fall away from its peak value like ω^2 at the lower frequencies. However, departures from rigid convection destroy the complete equivalence of the two spectra, a feature that introduces a finite level at zero frequency. At finite Mach numbers, the frequency spectrum is likely to retain most of its incompressible flow characteristics, but the wave-number spectrum is changed more dramatically. The wave-number spectrum then asymptotes to a constant level proportional to the square of mean-stream Mach number, and that level is maintained up to a wave-number k_0, of the order of M/δ^*. In that régime compressible-flow effects are dominant in setting the form of the spectrum.

3. The response of a large homogeneous plate to boundary-layer noise

The response of a large panel is governed by the equation of panel motion

$$-m\left\{\frac{\partial^2 y}{\partial t^2}+\beta\frac{\partial y}{\partial t}+\frac{B}{m}\nabla^4 y\right\} = p. \tag{3.0}$$

m is written for the mass of the panel per unit surface area, y is the panel displacement in a direction taking the panel into the turbulent flow, β is a mechanical damping coefficient and B is the bending stiffness. p is written for the total applied stress, incorporating not only the turbulent pressure field acting on one side of a rigid panel, but also the fluid loading due to panel response, known to be additive to the rigid-surface stress whenever the vibration is of low amplitude, provided, of course, the turbulent motion is unaffected by that of the panel.

The response equation can be expressed in terms of the generalized Fourier transform of the vibration velocity $U(\mathbf{k},\omega)$ and of the pressure exerted on a rigid surface $P_{\text{rigid}}(\mathbf{k},\omega)$, when one incorporates into the pressure field the fluid loading. The transform of the applied stress $P(\mathbf{k},\omega)$ can be related to $P_{\text{rigid}}(\mathbf{k},\omega)$ and $U(\mathbf{k},\omega)$ by a relation essentially expressing the superposition of pressures due to motion over a rigid surface and those due to surface response at low amplitude

$$P(\mathbf{k},\omega) = P_{\text{rigid}}(\mathbf{k},\omega)+\frac{\alpha\rho a_0 U(\mathbf{k},\omega)}{\{1-(ka_0/\omega)^2\}^{\frac{1}{2}}}; \tag{3.1}$$

α is either one or two, depending on whether only one side of the panel or both are exposed to the vibratory fluid loading.

The response power spectrum is derived by Fourier transforming equation (3.0) and multiplying the result by its complex conjugate. Use of equation (3.1) allows the spectra to be related through an equation which has two characteristic régimes depending on whether ka_0/ω is less or greater than unity. The first is the high-frequency régime where compressibility effects are important, and the vibration is able to lose energy to a vibration-excited sound field. The fluid loading then appears as an additional damping term in the equation relating the power spectra of response velocity $U^*(\mathbf{k},\omega)$ to that of the pressure field operative on a rigid surface $P^*_{\text{rigid}}(\mathbf{k},\omega)$. The equation is essentially that given by Corcos & Liepmann (1955) in their account of noise transmission through structures excited by a turbulent boundary layer when $|\omega| > a_0|\mathbf{k}|$,

$$U^*(\mathbf{k},\omega) = P^*_{\text{rigid}}(\mathbf{k},\omega)\frac{1}{(\omega m)^2}\left\{\left(1-\frac{k^4}{k_p^4}\right)^2+\left[\eta\frac{k^4}{k_p^4}\frac{\alpha\rho a_0}{\omega m}\left\{1-\left(\frac{ka_0}{\omega}\right)^2\right\}^{-\frac{1}{2}}\right]^2\right\}^{-1}. \tag{3.2}$$

k_p is the wave-number of free flexural waves and η is the mechanical loss factor. η is related to β by the relation

$$\omega\beta = \eta(B/m)\,k^4. \tag{3.3}$$

At lower frequencies the panel is unable to communicate its vibrational energy to the sound field, so that the fluid loading changes its character and becomes non-dissipative. The spectral equation is then of a different form

$$U^*(\mathbf{k},\omega)_{|\omega|<a_0|\mathbf{k}|} = P^*_{\text{rigid}}(\mathbf{k},\omega)\frac{1}{(\omega m)^2}\left\{\left[\left(1-\frac{k^4}{k_p^4}\right)-\frac{\alpha\rho a_0}{\omega m}\left\{\left(\frac{ka_0}{\omega}\right)^2-1\right\}^{-\frac{1}{2}}\right]^2+\eta^2\frac{k^8}{k_p^8}\right\}^{-1}. \tag{3.4}$$

Evidently, in the régime where incompressible arguments are largely valid, the fluid loading induced by vibration merely lowers the free-wave wave-number. The fluid loading remains, but is of a type completely unaffected by compressibility. That limit can be written down very simply, so that the low-frequency equation describing the response of a large homogeneous panel excited by boundary-layer turbulence is

$$U^*(\mathbf{k},\omega)_{|\omega|\ll a_0|\mathbf{k}|} = P^*_{\text{rigid}}(\mathbf{k},\omega)\frac{1}{(\omega m)^2}\left\{\left[\left(1-\frac{k^4}{k_p^4}\right)-\frac{\alpha\rho}{km}\right]^2+\eta^2\frac{k^8}{k_p^8}\right\}^{-1}. \tag{3.5}$$

The mechanical damping η, plays an essential part in controlling the resonant response. In this régime equation (2.15) provides the basis for estimating the forcing spectrum which would display features illustrated in figure 2(*b*). There would be no forcing field at low wave number and consequently there would be no response.

At higher frequencies equation (3.2) provides the proper description of response, asymptoting at sufficiently high frequency to the simpler form

$$U^*(\mathbf{k},\omega)_{|\omega|\gg a_0|\mathbf{k}|} = P^*_{\text{rigid}}(\mathbf{k},\omega)\frac{1}{(\omega m)^2}\left\{\left(1-\frac{k^4}{k_p^4}\right)^2+\left(\eta\frac{k^4}{k_p^4}+\frac{\alpha\rho a_0}{\omega m}\right)^2\right\}^{-1}. \tag{3.6}$$

The fluid loading here is strongly Mach-number dependent and is essentially unrelated to that of the incompressible-flow example, appearing only as an energy sink where the vibrational energy can be disposed of in an acoustic field. The most intense vibration will occur at the free-wave, or resonant, condition, where $k = k_p$. That is clearly the situation of most practical interest. k_p is related to the frequency ω through the bending stiffness, or more simply through the wave velocity c_p

$$\omega = k_p^2\,(B/m)^{\frac{1}{2}} = k_p c_p, \tag{3.7}$$

so that the free-wave response in the compressible-flow régime, which is the case for all free waves with supersonic phase velocities, is governed by the relatively simple equation relating the resonant response spectrum to that of the boundary-layer-pressure fluctuations induced on a rigid surface

$$U^*(\mathbf{k}_p, k_p c_p) = P^*_{\text{rigid}}(\mathbf{k}_p, k_p c_p)\frac{1}{m^2k_p^2c_p^2}\left\{\eta+\frac{\alpha\rho a_0}{mk_p c_p}\right\}^{-2}, \tag{3.8}$$

$$\omega \gg a_0|\mathbf{k}| \quad \text{or} \quad c_p \gg a_0.$$

In this régime the forcing pressure field is described by equation (2.11) so that the spectrum will be of the type sketched in figure 2(*a*), with a level proportional to the square of frequency near the frequency origin. The response spectrum will consequently be constant at low enough frequency. The level in this régime is that important in determining sound radiation and transmission by structural vibration, and is seen to be entirely dependent on effects directly attributable to the compressibility of the fluid. This is so for both the boundary-layer induced pressures and response, and represents a severe restriction of the usefulness of incompressible-flow arguments in the flow-noise problem.

We conclude with a comment on the régimes where the idealized large-plate example is likely to be of practical significance. The first section, dealing with the pressure field exerted on a rigid boundary by turbulent flow at finite Mach number, is likely to be relevant whenever the boundary layer is homogeneous in planes parallel to the surface, over lengths significantly larger than the acoustic wave-lengths of interest. However, the incompressible-flow models, in the frequency régimes where they are relevant, are likely to be valid whenever the flow is homogeneous over a length large compared with the boundary-layer thickness, a considerably less stringent dimensional requirement. In this second section, concerned more with the structural response, there are again two restrictions on surface dimensions. In the first place, for the structural dynamics to be similar to those of the infinite panel, the panel dimension must be large in comparison to the wave-lengths of free waves, and the dissipation must be high enough to inhibit modal response. In the second place, in order that the acoustic loading is that of the infinite plate, a panel must exceed by a large factor the acoustic wave-lengths of interest. The example is thus of restricted practical utility at low Mach number, but as Mach number and frequencies are increased the example must become more relevant. However, even at the lowest of Mach numbers, one point that has attracted considerable attention is clarified. That concerns the instantaneous force applied to a large plane surface by boundary-layer flow. It has been demonstrated here that the incompressible-flow arguments that predicted the force to vanish give only the limit of the more general result, that the force is proportional to the square of mean-flow Mach number.

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